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A MODIFIED CRITERION OF INSTABILITY OF MOTION

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Ideas related to Liapunov's second method and developed by Chetaev [e. g. see [1 and 3]) are used to obtain an instability criterion somewhat different from the well-known criteria of Liapunov and Chetaev.

1. Introduction. Let us consider the stability of the equilibrium position of a system of ordinary differential equations

$$dx \,/\, dt = X(x,\, t), \quad X(0,\, t) = 0 \tag{1.1}$$

We assume that the n-dimensional vector function $X(x,\, t)$ is continuous in t and has continuous partial derivatives with respect to x.

The criterion which we shall formulate is valid for systems of any order m; in order to illustrate the basic ideas geometrically, we shall first sonsider a simple example of a third-order system.

Let us assume that for $\xi > 0$ (here and below ξ, η, ζ represent the components of the vector x) we have the inequalities

$$\frac{d\xi}{dt} \geqslant \delta(\xi) > 0, \qquad \text{if} \quad \max(|\eta|, |\zeta|) \leqslant k\xi \tag{1.2}$$

$$\frac{d}{dt}(|\eta| - k\xi) < 0 \quad \text{for} \quad |\eta| = k\xi, \ |\zeta| \leqslant k\xi \tag{1.3}$$

$$\frac{d}{dt}(|\zeta| - k\xi) > 0 \quad \text{for} \quad |\zeta| = k\xi, \ |\eta| \leqslant k\xi \tag{1.4}$$

Let us consider the pyramid $OA'B'C'D'$ (see Fig. 1) defined by the inequality max $(|\eta|, |\zeta|) \leqslant k\xi$ and intersected by the plane $ABCD (\xi = \varepsilon)$ in the immediate neighborhood of the origin. Conditions (1.2)–(1.4) imply that the trajectories enter the truncated pyramid T, intersecting its surface at points belonging to the portion S_1 consisting of the faces $ABCD$, $BB'C'C$ and $DD'A'A$; similarly, the trajectories emerge from the pyramid through points of the portion S_2 of its surface consisting of the faces $BB'A'A$, $DD'C'C$ and $A'B'C'D'$. As we see from condition (1.2), the representing point can lie inside T for a finite time only; this implies that the family of trajectories entering the pyra-

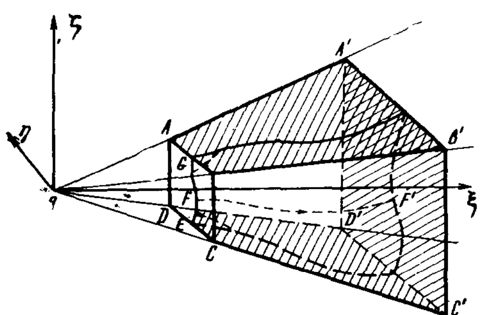

Fig. 1

mid T at the instant t_0 effects continuous mapping of the part S_1 of its surface onto the part S_2; each point of the closed broken line $ABB'C'C\,DD'A'A$ is mapped into itself.

Let us consider the mapping of the curve EFG. The corresponding curve $EF'G$ must be a continuous line lying entirely on the surface S_2 and connecting the points E and G. It must therefore contain points belonging to the face $A'B'C'D'$. This fact implies instability, since the face $ABCD$ can be placed arbitrarily close to the origin, and since there are always trajectories (such as FF') along which the representing point will reach the face $A'B'C'D'$ in a finite time.

2. Formulation and analysis of the problem. The criterion stated below generalizes the above basic idea. We hope that this generalization is not so recondite as to hinder its practical use (although it is possible that there may be another more convenient formulation).

Let us consider the set of points x belonging to a prescribed (small but finite) neighborhood Ω of the origin. We assume that there exist three single-valued functions $U(x, t)$, $V(x, t)$, and $W(x, t)$ such that for each sufficiently small $\varepsilon > 0$ there exists a closed domain Ω_ε and a set $S_{\varepsilon, t}$ nonempty for $t \geqslant t_0$ and defined as follows:

$$(\Omega_\varepsilon) \quad (x, t) \in \Omega_\varepsilon, \quad \text{if} \quad U \geqslant \varepsilon \quad \text{and} \quad \max(V, W) \leqslant 0$$

$$(S_{\varepsilon, t}) \quad x \in S_{\varepsilon, t}, \quad \text{if} \quad U = \varepsilon \quad \text{and} \quad \max(V, W) \leqslant 0$$

We assume that the function U is continuous in the domain Ω for $t \geqslant t_0$ and that it has an infinitely small upper bound; all that we require of the functions V and W (which may also depend on ε) is that they be continuous in some neighborhood of the domain Ω_ε; the values of $V(0, t)$ and $W(0,t)$, need not be known. We assume that the time derivatives U', V' and W' taken along the trajectories of Eqs.(1.1) are piecewise-continuous in the indicated neighborhood of the domain Ω_ε. Finally, we assume that the functions U, V, W in the domain Ω_ε for $t \geqslant t_0$ have the following properties (fulfilment of either Conditions a or Conditions b is sufficient):

1. $$U' \geqslant \delta(\varepsilon, t) \geqslant 0, \qquad \int_{t_0}^{\infty} \delta(\varepsilon, t)\, dt > M > 0 \quad (M = \text{const})$$

Here the constant M is independent of ε.

2a. $V' > 0$ for $V = 0$ and $W \leqslant 0$. (If $V' \to +0$ as $V \to -0$, then we can take a function $V_1 = V + \varepsilon_1$, $\varepsilon_1 > 0$, in which case $V_1' > 0$ for $V_1 = 0$. If the surface $V = 0$ is adjacent to the discontinuity surface V', then it is possible to alter the function V in suitable fashion.)

2b. $V' \geqslant -\gamma |V|$ as $V \to -0$, and $W \leqslant 0$ $(\gamma = \text{const})$

3a. $W' \leqslant \gamma |W|$ as $W \to -0$ and $V \leqslant 0$ $(\gamma = \text{const})$

3b. $\dot{W}' < 0$, as $W = 0$ and $V \leqslant 0$

The conditions which follow are usually fulfilled in practical problems. Let us assume that there exists a continuous mapping $y = f(x, t)$ of the domain Ω_ε into n-dimensional $(n < m)$ Euclidean space such that

4a. $V(x, t) < 0$, if $(x, t) \in \Omega_\varepsilon$, $y = f(x, t) = 0$

4b. $W(x, t) < 0$, if $(x, t) \in \Omega_\varepsilon$, $y = f(x, t) = 0$

5. There exists a closed connected subset $R_{\varepsilon,t} \subseteq S_{\varepsilon,t}$ homeomorphic to its mapping

$Q_{\varepsilon t} = f(R_{\varepsilon t}, t)$. The correspondence between the sets $R_{\varepsilon,t}$ and $Q_{\varepsilon,t}$ is given by Eq.

$$x = g_{\varepsilon,t}(y) = g_{\varepsilon,t}(f(x, t))$$

6a. The set $R_{\varepsilon,t}$ can be chosen in such a way that if $x \in R_{\varepsilon,t}$, then the limit equation $\varepsilon \to 0$ implies that $|x| \to 0$.

6b. If $x \in S_{\varepsilon,t}$, then as $\varepsilon \to 0$ we have the limit equation $|x| \to 0$.

7a. The equation $V(g_{\varepsilon,t}(y), t) = 0$ is fulfilled at the points of the boundary $P_{\varepsilon,t}$ of the set $Q_{\varepsilon,t}$.

7b. The equation $W(g_{\varepsilon,t}(y), t) = 0$ is fulfilled at the points of the boundary $P_{\varepsilon,t}$ of the set $Q_{\varepsilon,t}$.

8. The point $y = 0$ is an interior point of the domain $Q_{\varepsilon,t}$.

Under these conditions the origin becomes an unstable singular point (if the domain Ω_ε breaks down into several isolated domains, then fulfilment of the above conditions for one of these domains is sufficient).

Let us prove instability given fulfilment of Condition a . Let the origin be stable at the initial instant t_0 which we can assume to be equal to zero without loss of generality. There then exists a positive number ε such that any trajectory beginning at the instant $t = 0$ at the point $x_0 \in R_{\varepsilon,0}$ remains in the domain Ω for an infinitely long time (Conditions 5 and 6a). On the other hand, the trajectory must leave the confinites of the domain Ω_ε after a finite time (Condition 1), intersecting the boundary of this domain at a single point $(\bar{x}_0, \bar{t}_0)$ where $V = 0$ (Conditions 2a and 3a). This defines the mapping $(\bar{x}_0, \bar{t}_0) = h_{\varepsilon,0}(x_0)$ of the domain $R_{\varepsilon,0}$ into the space (x,t); this mapping is continuous by virtue of the continuity of the initial equations and of Condition 2a.

Now let us determine the continuous mapping of the domain $Q_{\varepsilon,0}$ onto the y-space

$$\bar{y} = f(\bar{x}_0, \bar{t}_0) = f(h_{\varepsilon,0}(x_0)) = f(h_{\varepsilon,0}(g_{\varepsilon,0}(y)) = F(y)$$

By Condition 7a we have

$$\bar{y} = F(y) = y, \quad \text{if } y \in P_{\varepsilon,0}$$

Simple topological considerations now indicate that the mapping $F(Q_{\varepsilon,0})$ covers all the interior points of the set $Q_{\varepsilon,0}$, including the point $\bar{y} = 0$ (Condition 8). (The domain of definition of the mapping F can be extended to the closed n-dimensional sphere $|y| \leqslant R$ containing the domain $Q_{\varepsilon,0}$; we achieve this by setting $F(y) = y$ for the points $y \notin Q_{\varepsilon,0}$. The assumption that the inequality $|F(y)| < \delta$ is not fulfilled at any point y implies that the projection $G(y) = -RF(y) / |F(y)|$ is continuous and has no stationary points; this contradicts the Brauer theorem). Since, by Condition 4a, $V(h_{\varepsilon,0}(g_{\varepsilon,0}(y)))< 0$ for $\bar{y} = 0$, we have a contradiction of our assumption that the singular point is stable. The theorem is proved.

The proof for Conditions b is similar to the above. Here we consider trajectories passing through the domain $\Omega_{\varepsilon,k}$ consisting of the points (x, t) for which $\varepsilon \leqslant U \leqslant k$ (ε is arbitrarily small) and $\max(V, W) \leqslant 0$. We choose a parameter $k < M$ so small that the inclusion $\Omega_{\varepsilon,k} \subset \Omega$ is valid.

Let us consider the motion of the representing point along the trajectory in the direction of decreasing t. Since the function U can have an infinitely small upper bound, the inequality $U(x, t) \geqslant k$ implies that $|x| \geqslant K(k) > 0$. On the other hand (see Condition 6b), the inequality $U(x, t) \leqslant \varepsilon$ implies that $|x| < \delta_1(\varepsilon)$, where δ_1 is arbitrarily small. For given k, ε and t_0 (e.g. for $t_0 = 0$) there always exists a T, $0 < T < \infty$ such that the sets $S_{k,T}$ and $S_{\varepsilon,0}$ are joined by at least one trajectory.

The value of T is bounded above; the upper bound is given by the inequality

$$\int_0^T \delta(\varepsilon, T)\,dt \leqslant k - \varepsilon$$

In proving this it is convenient to assume that $X(x, t) = X(x, -t)$ for $t < 0$.

Instability criteria similar to the Chetaev two-function theorem ([3], p. 225) are clearly derivable as special cases of the above criterion; to obtain them we need merely set either the function W (Case a) or the function V (Case b) equal to negative constants. On the other hand, the two-function theorem remains valid under other, less restrictive conditions. We can formulate the two following instability criteria:

c. Let us assume that $W = \text{const} < 0$; instability then follows from Conditions 1, 2b and 6b.

In fact, under these conditions any trajectory which reaches the set $S_{k,T}$ for $V < 0$ enters the domain Ω_ε only by way of the points of the set $S_{\varepsilon,\tau}$, $\tau < T$.

By making T vary between zero and the indicated upper bound, we can ensure that $\tau = 0$.

d. Let us assume that $V = \text{const} < 0$; instability then follows from Conditions 1, 3a and 6d, the latter requires the existence of an $x_0(\varepsilon) \in S_{\varepsilon,0}$ such that $W(x_0, 0) < 0$ and $|x_0(\varepsilon)| \to 0$ along with ε.

Under these conditions any trajectory which enters the domain Ω_ε at the point $(x_0, 0)$ cannot leave this domain in any other way save through the boundary of the domain Ω.

The validity of Criteria a and d is not contingent on the function U having an infinitely small upper bound. All that is required is that this function be uniformly bounded in the domain Ω and that it satisfy the Condition $U(0, t) = 0$. Furthermore, two-function Criterion b ($V = \text{const} < 0$) is wholly contained within Criterion b; the same cannot be said of Criteria a and c. The conditions of validity of Criterion c are fewer in number and less restrictive (especially as regards the behavior of V'); however, restrictions 6b on the set $S_{\varepsilon,t}$ and the restrictions on the function U (the infinitely small upper bound) are stronger than the corresponding conditions of Criterion a ($W = \text{const} < 0$).

The above formulation of the instability criterion is more constructive as regards the boundary restrictions and the restrictions on the function U' than is the classical theorem of Chetaev ([3], p. 225); this fact can have practical significance. Thus, if we interpret the original theorem of Chetaev ([3], p. 217) and his well-known single-function theorem as special cases of the two-function theorem ([3], p. 224), then we must admit the possibility of the functions U' and W' vanishing at the boundary $W = 0$. It is essential here that our Condition 1 be fulfilled (or that its fulfilment be possible upon suitable alteration of the definition of a function W dependent on ε).

3. Example. Let us investigate the stability of the equilibrium position in the following simple problem [4]. Let us take differential equations of the form

$$x_s' = \lambda_s x_s + X_s^{(2)}(x) + R_s(x, t) \qquad (s = 1, \ldots, 6) \tag{3.1}$$

We assume that the following equations are fulfilled:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0,\ \lambda_4^* = -\dot{\lambda}_4 = \lambda_5,\ x_5 = x_4^*,\ \lambda_6 = \lambda_6^* < 0 \tag{3.2}$$

The symbols $X_s^{(2)}(x)$ represent homogeneous second-degree polynomials with constant coefficients; the functions R_s satisfy the inequalities

$$|R_s(x, t)| \leqslant k|x|^3, \quad k = \text{const} < \infty \quad \text{for } t > 0 \tag{3.3}$$

The complex conjugate "critical" variables x_4, x_5 have been introduced to illustrate the special features associated with purely imaginary eigenvalues. There can be many pairs of such eigenvalues. If all these pairs are distinct, no additional difficulties arise. All terms except x_1x_4, x_2x_4 or x_3x_4 can be eliminated from the expression for $X_4^{(2)}$; we shall assume that this has already been done [4].

Similarly, terms of the type x_1x_4, x_1x_5, x_2x_4, etc., can be eliminated from the expressions for $X_1^{(2)}$, $X_2^{(2)}$, $X_3^{(2)}$. The "noncritical" variable x_6 has been introduced in order that our system might have an eigenvalue with a negative real part; increasing the number of such eigenvalues entails no difficulties, even if some of the eigenvalues are not distinct. All the terms in the expressions for $X_s^{(2)}$, $s = 1,\ldots, 5$ which contain x_6 can be readily eliminated (e.g. see [4]); we shall assume that this has already been done.

We begin the analysis of the prblem by determining the direction of the instability radius for which $\rho = O(|x_1|)$; if such a radius does not exist, then we shall attempt to find that for which $\rho = O(|x_2|)$, $x_1 = o(\rho)$, etc.

Let us set

$$y_2 = x_2 - \alpha x_1, \quad y_3 = x_3 - \beta x_1, \; y_s = x_s \quad (s = 1, 4, 5, 6) \tag{3.4}$$

and choose the constants α and β in such a way as to exclude the terms containing y_1^2 from the expressions for y_2' and y_3'. These constants can be determined by solving the two equations for α and β; one of these constants occurs in the second power, the other in the third power. The system of equations can be readily solved by the graphic method. Computational difficulties arise in the case of more than three critical variables associated with zero eigenvalues.

Let us assume that there exists at least one pair of constants α, β satisfying the above conditions. The initial equations then become

$$y_1' = ay_1^2 + p_1(y, t), \quad \begin{pmatrix} y_2' \\ y_3' \end{pmatrix} = y_1 A \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} p_2(y, t) \\ p_3(y, t) \end{pmatrix} \tag{3.5}$$

$$\frac{d}{dt}(y_4y_5) = 2by_1y_4y_5 + p_4(y, t), \qquad y_6' = \lambda_6 y_6 + p_6(y, t)$$

where the quantities a and b are real, and where the functions p_1, p_2, p_3, p_4, p_6 satisfy the inequalities

$$\begin{gathered} |p_1| \leqslant k_1|y|(|y_2| + |y_3| + |y_4|) + k_2|y|^3 \\ |p_2|, \; |p_3| \leqslant k_3(|y_2|^2 + |y_3|^2 + |y_4|^2) + k_4|y|^3 \\ |p_4| \leqslant k_5 y_4 y_5 (|y_2| + |y_3|) + k_6|y|^3|y_4| \\ |p_6| \leqslant k_7|y|^2 \end{gathered} \tag{3.6}$$

Let us assume that $a \neq 0$; we can also assume without loss of generality that $a > 0$. The matrix A (with the eigenvalues μ_2 and μ_3) is transformable to normal Jordan form; here $y_3 = y_2^*$ if $\mu_3 = \mu_2^* \neq \mu_2$. (If $a = 0$, then instability can in some cases be established by investigating the terms proportional to y_1^3. Analysis of the case $a \to 0$, at least in the case of an autonomous system when the functions R_s in Eqs. (3.1) do not explicitly depend on t, is complicated by the existence of a singular point $y_1 \sim a$, y_2, $y_3 \sim a^2$, y_4, y_5, $y_6 \approx 0$, which tends to the origin).

If the above conditions are fulfilled, then the origin turns out to be an unstable singular point of Eqs. (3.1) (*).

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*) The reviewer of the present paper has noted that this example can generally be analyzed by the method of Kamenkov [5]. I believe, however, that the method described here is simpler. It also has the advantage of being free of the somewhat artificial

Let us prove this by applying the above criterion.

We take a constant γ subject to the inequality

$$0<\gamma<1 \tag{3.7}$$

(further restrictions on the choice of γ will be noted below).

Next we consider a small neighborhood of the origin and set

$$U=y_1, \quad Z_s=y_s y_s^* - c_s^2 y_1^{2+2\gamma} \quad (s=2,\ldots,6) \tag{3.8}$$

The positive constants c_s can all be set equal to unity (except in Case III below: see Formula (3.23)).

We denote the values of Z'_s at the points where $Z_s=0$ by $Z'_{s,0}$. We then define the domain Ω_ε by stipulating that

$$(\Omega_\varepsilon) \quad x\in\Omega_\varepsilon, \quad \text{e if } U\geqslant\varepsilon \text{ and } \max_s Z_s\leqslant 0 \tag{3.9}$$

For the points of this domain we have (see (3.5), (3.6))

$$U'=ay_1^2+O(y_1^{2+\gamma}+y_1^3) \tag{3.10}$$

$$Z'_{4,0}=Z'_{5,0}=2d_4c_4^2y_1^{3+2\gamma}+O(y_1^{3+3\gamma}+y_1^{4+\gamma}) \tag{3.11}$$

$$d_4=d_5=b-a(1+\gamma) \tag{3.12}$$

Moreover,

$$Z'_{6,0}=2\lambda_6c_6^2y_1^{2+2\gamma}+O(y_1^{3+\gamma}) \tag{3.13}$$

where by (3.2)

$$\lambda_6<0 \tag{3.14}$$

To compute $Z'_{2,0}$ and $Z'_{3,0}$ we consider the possible types of matrix A:

I. The eigenvalues are distinct and complex-conjugate,

$$A=\begin{pmatrix}\mu_2 & 0\\ 0 & \mu_2^*\end{pmatrix}, \qquad Z'_{2,0}=Z'_{3,0}=2d_2c_2^2y_1^{3+2\gamma}+O(y_1^{3+3\gamma}+y_1^{4+\gamma}) \tag{3.15}$$

$$d_2=d_3=\operatorname{Re}\mu_2-a(1+\gamma) \tag{3.16}$$

II. The eigenvalues are real and distinct. The matrix A is diagonal,

$$A=\begin{pmatrix}\mu_2 & 0\\ 0 & \mu_3\end{pmatrix}, \qquad \begin{aligned}Z'_{2,0}&=2d_2c_2^2y_1^{3+2\gamma}+O(y_1^{3+3\gamma}+y_1^{4+\gamma})\\ Z'_{3,0}&=2d_3c_3^2y_1^{3+2\gamma}+O(y_1^{3+3\gamma}+y_1^{4+\gamma})\end{aligned} \tag{3.17}$$

$$d_2=\mu_2-a(1+\gamma), \qquad d_3=\mu_3-a(1+\gamma) \tag{3.18}$$

III. The eigenvalues are real and distinct. The matrix A is nondiagonal,

$$A=\begin{pmatrix}\mu_2 & 0\\ 1 & \mu_2\end{pmatrix}, \qquad \begin{aligned}Z'_{2,0}&=2d_2c_2^2y_1^{3+2\gamma}+O(y_1^{3+3\gamma}+y_1^{4+\gamma})\\ Z'_{3,0}&=2d_3c_3^2y_1^{3+2\gamma}+O(y_1^{3+3\gamma}+y_1^{4+\gamma})\end{aligned} \tag{3.19}$$

In this case d_2 is defined as in (3.18); the quantity d_3 is no longer constant, and depends on y_2 and sgn y_3. Setting

$$|y_2|=\theta c_2y_1^{1+\gamma}, \qquad |\theta|\leqslant 1 \tag{3.20}$$

we obtain

$$d_3=\mu_2-a(1+\gamma)\pm\theta c_2/c_3 \tag{3.21}$$

Now let us choose the constant γ in such a way that none of the quantities d_s equals

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(cont. from opposite page) restrictions on the time dependence of the higher-order terms R (see (3.1)) involved in Kamenkov's method (e.g. see [5], pp. 123-127).

zero. Specifically, we require that

$$|d_s| \gtrsim a / 2p \tag{3.22}$$

in which p is the number of critical variables in the problem. In Case III we must also make the ratio c_2 / c_3 sufficiently small, e.g.

$$c_2 / c_3 \leqslant a / 8 \tag{3.23}$$

It is convenient in practice to place γ midway between zero and unity.

Let us break down the quantities Z_s into two groups according to whether the functions $Z'_{s,0}$ are positive or negative. We set

$$V = \max_s Z_s, \quad (Z'_{s,0} > 0), \qquad W = \max_s Z_s \quad (Z'_{s,0} < 0) \tag{3.24}$$

If $Z'_{s,0} < 0$ for all s, then instability follows immediately from the two-function theorem of Chetaev. Otherwise we must apply the above criterion, whose conditions are clearly fulfilled. (The mapping into n-dimensional space is effected by projecting onto the half-space those y_s for which $Z'_{s,0} > 0$: it is sufficient to take the real and imaginary parts of each pair of complex-conjugate coordinates. The subset $R_{\varepsilon,t}$ can be defined by setting $y_s = 0$ for the s such that $Z'_{s,0} < 0$).

Even though the origin is unstable for most systems of this type (Eqs. (3.1) – (3.3)), it is easy to construct cases when the method does not apply and the problem of stability must be solved by considering third-order terms. For example, let us consider system

$$x'_1 = x_1x_2 + x_1x_3 \pm x_1^3, \quad x_2' = -x_1^2 \pm x_2^3, \quad x_3' = -x_1^3 \pm x_3^3 \tag{3.25}$$

for which

$$\frac{d}{dt}(x_1^2 + x_2^2 + x_3^2) = \pm 2(x_1^4 \pm x_2^4 + x_3^4) \tag{3.26}$$

The upper sign in this equation corresponds to instability; the lower sign corresponds to stability.

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